TOPOLOGICAL INVARIANTS FOR SEMIGROUPS OF HOLOMORPHIC SELF-MAPS OF THE UNIT DISC

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RÉSUMÉ. Soient (φ_t), (ϕ_t) deux semi-groupes à un paramètre d'endomorphismes holomorphes du disque unité $\mathbb{D} \subset \mathbb{C}$. Soit $f : \mathbb{D} \to \mathbb{D}$ un homéomorphisme. Nous montrons que si $f \circ \phi_t = \varphi_t \circ f$ pour tout $t \geq 0$, alors f s'étend à un homéomorphisme de $\overline{\mathbb{D}}$ en dehors des arcs de contact exceptionnels maximaux (en particulier, si l'on considère des semi-groupes elliptiques, f s'étend toujours à un homomorphisme de $\overline{\mathbb{D}}$). En utilisant ce résultat, nous étudions les invariants topologiques pour les semi-groupes à un paramètre d'endomorphismes holomorphes du disque unité.

ABSTRACT. Let (φ_t), (ϕ_t) be two one-parameter semigroups of holomorphic self-maps of the unit disc $\mathbb{D} \subset \mathbb{C}$. Let $f : \mathbb{D} \to \mathbb{D}$ be a homeomorphism. We prove that, if $f \circ \phi_t = \varphi_t \circ f$ for all $t \geq 0$, then f extends to a homeomorphism of $\overline{\mathbb{D}}$ outside exceptional maximal contact arcs (in particular, for elliptic semigroups, f extends to a homeomorphism of $\overline{\mathbb{D}}$). Using this result, we study topological invariants for one-parameter semigroups of holomorphic self-maps of the unit disc.

CONTENTS

1. Introduction

One-parameter continuous semigroups of holomorphic self-maps of $\mathbb{D}$ (sometimes just called holomorphic semigroups in $\mathbb{D}$ for short) have been widely studied, see, *e.g.*, [3, 1, 21]. In recent years, particular attention was paid to the boundary behavior of semigroups in $\mathbb{D}$, their boundary trajectories, boundary singularities, and fine properties of semigroups have been discovered and investigated via the associated infinitesimal generators and Kœnigs functions, see, *e.g.*, [5, 8, 9, 10, 6, 18, 14, 15, 16, 17] and references therein. For instance, the dynamical type (elliptic, hyperbolic, parabolic), the hyperbolic step, the Denjoy – Wolff point, boundary (regular or super-repelling) fixed points, regular poles, fractional singularities, maximal contact arcs, can be well understood in terms of the geometry of the image of the Kœnigs function and the analytic properties of the infinitesimal generator.

All these objects are *holomorphic invariants* of a semigroup of holomorphic self-maps. Namely, if two semigroups of holomorphic self-maps are holomorphically conjugated through an automorphism of the unit disc, there is a one-to-one correspondence among the objects listed before and the way they are displaced along the boundary of the disc. This follows easily from the fact that automorphisms of the unit disc are linear fractional maps. Therefore, holomorphic invariants form a huge family and each class of holomorphic conjugations is relatively small. One might expect that lowering the regularity of the conjugation map, the number of invariants decreases.

In this paper we are interested in studying *topological invariants* of semigroups of holomorphic self-maps of the unit disc. Namely, we consider properties of holomorphic semigroups which are invariant under conjugation via homeomorphisms of the unit disc, *without any assumption on the regularity of the conjugacy map on the boundary of the unit disc.* One might expect that all holomorphic invariants related to the boundary behavior are destroyed. However, and quite surprisingly, most of them survive and are topological invariants. Roughly speaking, the holomorphic invariants which are related to the isometric (with respect to the Poincaré metric) nature of holomorphic conjugacies are destroyed under topological conjugation, but, those invariants which are related to

the dynamics survive, with the exception of some special maximal contact arcs, which we call *exceptional maximal arcs*.

To be more precise, let us fix some notations. We refer the reader to the next sections for the corresponding definitions. Let (φ_t) be a (continuous 1-parameter) semigroup of holomorphic self-maps of the unit disc $\mathbb{D}$. If (φ_t) is not a group of elliptic automorphisms, there exists a unique point $x \in \overline{\mathbb{D}}$, called the Denjoy–Wolff point of (φ_t), such that $\lim_{t\to\infty} \varphi_t(z) = x$ for all $z \in \mathbb{D}$. The semigroup (φ_t) is called *elliptic* if its Denjoy–Wolff point belongs to $\mathbb{D}$. If (φ_t) is non-elliptic, it is called *parabolic* if for every $t \geq 0$ the non-tangential limit of φ_t' at its Denjoy–Wolff point is 1, otherwise it is called *hyperbolic*.

While it is clear that elliptic and non-elliptic semigroups of holomorphic self-maps of $\mathbb{D}$ cannot be topologically conjugated, we first prove in Corollary 3.6 that every non-elliptic semigroup of holomorphic self-maps of $\mathbb{D}$ is topologically conjugated to a hyperbolic semigroup of holomorphic self-maps of $\mathbb{D}$.

The basic tool for this result and the next ones, is provided by the existence of universal holomorphic models for semigroups of holomorphic self-maps, as introduced in [12, 2], and the characterization of topological conjugation between two semigroups via topological conjugation of their holomorphic models (see Section 3).

In [18], P. Gumenyuk proved that for every $x \in \partial\mathbb{D}$ and for every $t \geq 0$, φ_t has non-tangential limit at x, so that $\varphi_t(x)$ is well-defined, and the curve $[0, +\infty] \ni t \mapsto \varphi_t(x) \in \overline{\mathbb{D}}$ is continuous. Therefore one can define the *life-time* $T(x)$ of $x \in \partial\mathbb{D}$ on the boundary to be the supremum of $t \in [0, +\infty]$ such that $\varphi_t(x) \in \partial\mathbb{D}$ (see Section 4). If $T(x) > 0$, it follows from [6] that either x is a fixed point of the semigroup (and in such a case $T(x) = \infty$) or x belongs to a *maximal contact arc*, that is, A is a non empty open arc in $\partial\mathbb{D}$ maximal with respect to the property that the infinitesimal generator of (φ_t) extends holomorphically through A, never vanishes and it is tangent to ∂A. If A is a maximal contact arc and $T(x) = \infty$, then we call A an *exceptional maximal contact arc*. In this case, the ending point of A (with respect to the natural orientation given by the curve $t \mapsto \varphi_t(x)$), is the Denjoy–Wolff point of (φ_t). In particular, if (φ_t) is elliptic, it does not admit exceptional maximal contact arcs. A semigroup has at most two exceptional maximal contact arcs.

In Proposition 4.7, we prove that the life-time is a continuous function on each maximal contact arc and we use it to characterize the starting point of a maximal contact arc.

In case $x \in \partial\mathbb{D}$ is a fixed point for (φ_t) which is super-repelling (or non-regular) we show, as a consequence of Proposition 4.9, that x is either the starting point of a maximal contact arc or there exists a backward orbit of (φ_t) landing at x.

Let us denote by $M(\varphi_t) \subset \partial\mathbb{D}$ the set of points which belong to an exceptional maximal contact arc for (φ_t) (note that $M(\varphi_t)$ is open and possibly empty). Let $x \in \overline{\mathbb{D}}$ be the Denjoy–Wolff point of (φ_t) and let

$$E(\varphi_t) := \overline{M(\varphi_t) \cup \{x\}} \cap \partial\mathbb{D}.$$

Note that $E(\varphi_t) = \emptyset$ if $x \in \mathbb{D}$ while $\{x\} \subseteq E(\varphi_t)$ if $x \in \partial\mathbb{D}$. We can now state the main result of this paper:

Theorem 1.1. *Let $(\varphi_t), (\phi_t)$ be two semigroups of holomorphic self-maps of $\mathbb{D}$, which are not elliptic groups. Let $x \in \overline{\mathbb{D}}$ be the Denjoy–Wolff point of (φ_t) and $y \in \overline{\mathbb{D}}$ the Denjoy–Wolff point of (ϕ_t). Assume that $f : \mathbb{D} \to \mathbb{D}$ is a homeomorphism (no regularity on $\partial\mathbb{D}$ is assumed) such that $f \circ \phi_t = \varphi_t \circ f$ for all $t \geq 0$. Then f extends to a homeomorphism from $\overline{\mathbb{D}} \setminus E(\phi_t)$ to $\overline{\mathbb{D}} \setminus E(\varphi_t)$.*

Moreover, for all $p \in \partial\mathbb{D} \setminus E(\phi_t)$ it holds $T(p) = T(f(p))$ and

$$f(\phi_t(p)) = \varphi_t(f(p)) \quad \text{for all } t \geq 0.$$

Finally, if $x \in \partial\mathbb{D}$, then $y \in \partial\mathbb{D}$ and $\angle \lim_{z \to x} f(z) = y$.

The previous theorem for the non-elliptic case is the content of Proposition 5.2 and Proposition 7.1. The way to adapt the proofs for the elliptic case is sketched in Section 8. The proof is based on the combined use of the universal holomorphic model and Carathéodory's prime ends topology.

With Theorem 1.1 at hands, it is quite easy to prove that most of the objects described before are topological invariants. These are described in Section 6. We summarize here such results:

Proposition 1.2. *The following are topological invariants for semigroups of holomorphic self-maps of $\mathbb{D}$:*

(1) *fixed points in $\mathbb{D}$,*

(2) *boundary regular fixed points which do not start exceptional maximal contact arcs,*

(3) *super-repelling fixed points which start maximal, non exceptional, contact arcs,*

(4) *super-repelling fixed points having backward orbits of (φ_t) landing at such points,*

(5) *maximal contact arcs which are not exceptional,*

(6) *life time on the boundary of a boundary point,*

(7) *boundary points of continuity for a semigroup, i.e., points on $\partial\mathbb{D}$ such that every element of the semigroup has unrestricted limits at such points.*

In Section 7 we study the behavior of topological conjugations on exceptional maximal contact arcs and the Denjoy–Wolff point. In particular, in Proposition 7.1 we show that the topological conjugacy map always has the non-tangential limit at a boundary regular fixed point which is the initial point of an exceptional maximal contact arc and such a limit is a boundary regular fixed point (possibly the Denjoy–Wolff point). However, as shown with several examples, the unrestricted limits at such points and at the Denjoy–Wolff point might fail to exist, and exceptional maximal contact arcs can be mapped entirely to the Denjoy–Wolff point.

Finally, in Section 8, we sketch how to extend these results to the case of elliptic semigroups.

2. Preliminaries

2.1. **Contact and fixed points.** For the unproven statements, we refer the reader to, *e.g.*, [1], [13] or [21].

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, $x \in \partial\mathbb{D}$, and let

$$\alpha_x(f) := \liminf_{z \to x} \frac{1 - |f(z)|}{1 - |z|}.$$

From Julia's lemma it follows that $\alpha_x(f) > 0$. The number $\alpha_x(f)$ is called the *boundary dilatation coefficient* of f.

If $f : \mathbb{D} \to \mathbb{C}$ is a map and $x \in \partial\mathbb{D}$, we write $\angle \lim_{z \to x} f(z)$ for the non-tangential (or angular) limit of f at x. If this limit exists and no confusion arises, we denote its value simply by $f(x)$.

Definition 2.1. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. A point $x \in \partial\mathbb{D}$ is said to be a *contact point* (respectively, *boundary fixed point*) of f, if $f(x) := \angle \lim_{z \to x} f(z)$ exists and belongs to $\partial\mathbb{D}$ (respectively, coincides with x). If in addition

$$(2.1) \qquad\qquad\qquad \alpha_f(x) < +\infty,$$

then x is called a *regular contact point* (respectively, *boundary regular fixed point*) of f.

A boundary fixed point which is not regular is said to be *super-repelling*.

Remark 2.2. By the Julia–Wolff–Carathéodory theorem, condition (2.1) in the above definition is sufficient on its own for $x \in \partial\mathbb{D}$ to be a regular contact point of f.

If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, neither the identity nor an elliptic automorphism, by the Denjoy–Wolff theorem, there exists a unique point $x \in \overline{\mathbb{D}}$, called the *Denjoy–Wolff point* of f, such that $f(x) = x$ and the sequence of iterates $\{f^{\circ k}\}$ converges uniformly on compacta of $\mathbb{D}$ to the constant map $z \mapsto x$. Moreover, if $x \in \partial\mathbb{D}$ then x is a boundary regular fixed point of f and $\alpha_f(x) \in (0, 1]$.

2.2. **Semigroups.** A (one-parameter) semigroup (ϕ_t) of holomorphic self-maps of $\mathbb{D}$ is a continuous homomorphism $t \mapsto \phi_t$ from the additive semigroup $(\mathbb{R}_{\geq 0}, +)$ of non-negative real numbers to the semigroup $(\mathsf{Hol}(\mathbb{D}, \mathbb{D}), \circ)$ of holomorphic self-maps of $\mathbb{D}$ with respect to composition, endowed with the topology of uniform convergence on compacta.

It is known that if (ϕ_t) is a semigroup of holomorphic self-maps of $\mathbb{D}$ and ϕ_{t_0} is an automorphism of $\mathbb{D}$ for some $t_0 > 0$, then (ϕ_t) can be extended to a group in $\mathsf{Aut}(\mathbb{D})$.

Definition 2.3. A *boundary (regular) fixed point* for a semigroup (ϕ_t) of holomorphic self-maps of $\mathbb{D}$ is a point $x \in \partial\mathbb{D}$ which is a boundary (regular) fixed point of ϕ_t for all $t > 0$.

Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$. It is well known that ϕ_{t_0} has a fixed point in $\mathbb{D}$ for some $t_0 > 0$ if and only if there exists $x \in \mathbb{D}$ such that $\phi_t(x) = x$ for all $t \geq 0$. In such a case, the semigroup is called *elliptic*. Moreover, there exists $\lambda \in \mathbb{C}$

with $\operatorname{Re}\lambda \leq 0$ such that $\phi'_t(x) = e^{\lambda t}$ for all $t \geq 0$. The elliptic semigroup (ϕ_t) is a group if and only if $\operatorname{Re}\lambda = 0$. The number λ is called the *spectral value* of the elliptic semigroup.

If the semigroup (ϕ_t) is not elliptic, then there exists a unique $x \in \partial\mathbb{D}$ which is the Denjoy–Wolff point of ϕ_t for all $t > 0$. Moreover, there exists $\lambda \leq 0$, the *dilation of* (ϕ_t), such that
$$\alpha_{\phi_t}(x) = e^{\lambda t} \quad t \geq 0.$$
A non-elliptic semigroup is said *hyperbolic* if its dilation is non-zero, while it is said *parabolic* if the dilation is 0.

It is also known (see, [10, Theorem 1], [11, Theorem 2], [22, pag. 255], [14]) that a point $x \in \partial\mathbb{D}$ is a boundary (regular) fixed point of ϕ_{t_0} for some $t_0 > 0$ if and only if it is a boundary (regular) fixed point of ϕ_t for all $t \geq 0$.

By Berkson and Porta's theorem [3, Theorem (1.1)], if (ϕ_t) is a semigroup in of holomorphic self-maps of $\mathbb{D}$, then $t \mapsto \phi_t(z)$ is real-analytic and there exists a unique holomorphic vector field $G : \mathbb{D} \to \mathbb{C}$ such that
$$\frac{\partial\phi_t(z)}{\partial t} = G(\phi_t(z)) \qquad \text{for all } z \in \mathbb{D} \text{ and all } t \geq 0.$$
This vector field G, called the *infinitesimal generator* of (ϕ_t), is *semicomplete* in the sense that the Cauchy problem
$$\begin{cases} \frac{dx(t)}{dt} = G(x(t)), \\ x(0) = z, \end{cases}$$
has a solution $x^z : [0, +\infty) \to \mathbb{D}$ for every $z \in \mathbb{D}$. Conversely, any semicomplete holomorphic vector field in $\mathbb{D}$ generates a semigroup of holomorphic self-maps of $\mathbb{D}$.

3. Topological and holomorphic models for semigroups

3.1. Semigroups and holomorphic models.

Definition 3.1. Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$. A *topological model* for (ϕ_t) is a triple (Ω, h, Φ_t) such that Ω is an open subset of $\mathbb{C}$, Φ_t is a group of (holomorphic) automorphisms of Ω and $h : \mathbb{D} \to h(\mathbb{D}) \subset \Omega$ is a homeomorphism on the image (that is, it is open, continuous and injective), $h \circ \phi_t = \Phi_t \circ h$ and

(3.1) $$\cup_{t\geq 0}\Phi_t^{-1}(h(\mathbb{D})) = \Omega.$$

A *holomorphic model* is a topological model (Ω, h, Φ_t) such that $h : \mathbb{D} \to \Omega$ is univalent.

The previous notion of holomorphic model was introduced in [2], where it was proved that every semigroup of holomorphic self-maps of any complex manifold admits a holomorphic model, unique up to "holomorphic equivalence". Moreover, a model is "universal" in the sense that every other conjugation of the semigroup to a group of automorphisms factorize through the model (see [2, Section 6] for more details).

Notice that given a topological model (Ω, h, Φ_t) for a semigroup (ϕ_t) of holomorphic self-maps of $\mathbb{D}$ then (ϕ_t) is a group if and only if $h(\mathbb{D}) = \Omega$.

Definition 3.2. Let (ϕ_t) and $(\tilde{\phi}_t)$ be two semigroups of holomorphic self-maps of $\mathbb{D}$. Let (Ω, h, Φ_t) be a holomorphic (respectively, topological) model for (ϕ_t), and $(\tilde{\Omega}, \tilde{h}, \tilde{\Phi}_t)$ a holomorphic (respectively, topological) model for $(\tilde{\phi}_t)$. We say that (Ω, h, Φ_t) and $(\tilde{\Omega}, \tilde{h}, \tilde{\Phi}_t)$ are *holomorphically (respect., topologically) conjugated* if there exists a biholomorphism (respect., homeomorphism) $\tau : \Omega \to \tilde{\Omega}$ such that $\tau \circ \Phi_t = \tilde{\Phi}_t \circ \tau$ and $\tau(h(\mathbb{D})) = \tilde{h}(\mathbb{D})$. The map τ is called a *holomorphic (respect., topological) conjugation of models*.

In case $(\phi_t) = (\tilde{\phi}_t)$ and $\tilde{h} = \tau \circ h$, we say that the models (Ω, h, Φ_t) and $(\tilde{\Omega}, \tilde{h}, \tilde{\Phi}_t)$ are holomorphically (respect., topologically) *equivalent*.

Conjugated semigroups correspond to conjugated models:

Proposition 3.3. *Let $(\phi_t), (\tilde{\phi}_t)$ be two semigroups of holomorphic self-maps of $\mathbb{D}$, with holomorphic (respectively, topological) models (Ω, h, Φ_t) and $(\tilde{\Omega}, \tilde{h}, \tilde{\Phi}_t)$. The following are equivalent:*

(1) *the holomorphic (respectively, topological) models (Ω, h, Φ_t) and $(\tilde{\Omega}, \tilde{h}, \tilde{\Phi}_t)$ are holomorphically (respect., topologically) conjugated;*

(2) *the semigroups (ϕ_t) and $(\tilde{\phi}_t)$ are holomorphically (respect., topologically) conjugated, that is, there exists an automorphism (respect., homeomorphism) $T : \mathbb{D} \to \mathbb{D}$ such that $T \circ \tilde{\phi}_t \circ T^{-1} = \phi_t$ for all $t \geq 0$.*

Proof. (1) implies (2). Let τ be a holomorphic (respect., topological) conjugation of models between (Ω, h, Φ_t) and $(\tilde{\Omega}, \tilde{h}, \tilde{\Phi}_t)$, that is, $\tau : \Omega \to \tilde{\Omega}$ is a biholomorphism (respect., homeomorphism) such that $\tau \circ \Phi_t = \tilde{\Phi}_t \circ \tau$ and $\tau(h(\mathbb{D})) = \tilde{h}(\mathbb{D})$. Define $T := \tilde{h}^{-1} \circ \tau \circ h$. It is easy to see that T is a holomorphic (respect., topological) conjugation between (ϕ_t) and $(\tilde{\phi}_t)$.

(2) implies (1). Let T be a holomorphic (respect., topological) conjugation between (ϕ_t) and $(\tilde{\phi}_t)$. Next, we extend $\tau := h^{-1} \circ T \circ \tilde{h}$ to a biholomorphism (respect., homeomorphism) $\tau : \Omega \to \tilde{\Omega}$ which intertwines Φ_t with $\tilde{\Phi}_t$ in the following way. Let $\Omega_t := \Phi_t^{-1}(h(\mathbb{D}))$ and $\tilde{\Omega}_t := \tilde{\Phi}_t^{-1}(\tilde{h}(\mathbb{D}))$, $t \geq 0$. Note that, if $t \geq s$ then $\Omega_s \subseteq \Omega_t$, since $\Phi_t = \Phi_{t-s} \circ \Phi_s$ and $h(\mathbb{D})$ is invariant under Φ_t for all $t \geq 0$. Moreover, $\Phi_t(\Omega_t) = h(\mathbb{D})$ and, by the definition of model, $\Omega = \cup_{t \geq 0} \Omega_t$. Similarly for $\tilde{\Omega}$.

For $t \geq 0$, define $\tau_t : \Omega_t \to \tilde{\Omega}_t$ by

$$\tau_t := \tilde{\Phi}_t^{-1} \circ \tilde{h} \circ T \circ h^{-1} \circ \Phi_t.$$

Clearly, τ_t is a biholomorphism (respect., homeomorphism) from Ω_t and $\tilde{\Omega}_t$. Also, by definition, $\tilde{\Phi}_t \circ \tau_0 = \tau_0 \circ \Phi_t|_{\Omega_0}$ for all $t \geq 0$.

Let $0 \leq s \leq t$ and let $\omega \in \Omega_s$. Then

$$\tau_t(\omega) = \tilde{\Phi}_t^{-1}(\tau_0(\Phi_{t-s}(\Phi_s(\omega)))) = \tilde{\Phi}_t^{-1}(\tilde{\Phi}_{t-s}(\tau_0(\Phi_s(\omega)))) = \tau_s(\omega).$$

Therefore, for $0 \leq s \leq t$ it holds $\tau_t|_{\Omega_s} = \tau_s$. Hence, the map $\tau : \Omega \to \tilde{\Omega}$ defined by $\tau|_{\Omega_t} := \tau_t$ is well defined and it is a biholomorphism (respect., homeomorphism) from Ω and $\tilde{\Omega}$.

Finally, by a similar argument as before, one can show that for all $t \geq 0$, $\tau_t \circ \Phi_s|_{\Omega_t} = \tilde{\Phi}_s \circ \tau_t$ for all $s \geq 0$, hence $\tau \circ \Phi_s = \tilde{\Phi}_s \circ \tau$ for all $s \geq 0$, concluding the proof. $\qquad\square$

Holomorphic models always exist and are unique up to holomorphic equivalence of models. In what follows we denote by $\mathbb{H} := \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\}$, $\mathbb{H}^- := \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta < 0\}$ and, given $\rho > 0$, $\mathbb{S}_\rho := \{\zeta \in \mathbb{C} : 0 < \operatorname{Re}\zeta < \rho\}$. We simply write $\mathbb{S} := \mathbb{S}_1$. The following result sums up the results in [2, 12], see also [1].

Theorem 3.4. *Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$.*

   (1) *If (ϕ_t) is a group of elliptic automorphism, then it has a holomorphic model $(\mathbb{D}, h, z \mapsto e^{i\theta t}z)$ for some $\theta \in (-\pi, \pi]$, with $h(\mathbb{D}) = \mathbb{D}$.*

   (2) *If (ϕ_t) is elliptic, not a group, then (ϕ_t) has a holomorphic model $(\mathbb{C}, h, z \mapsto e^{\lambda t}z)$, for some $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda < 0$.*

   (3) *If (ϕ_t) is non-elliptic, then (ϕ_t) has a holomorphic model (Ω, h, Φ_t) where $\Omega = \mathbb{C}, \mathbb{H}, \mathbb{H}^-, \mathbb{S}_\rho$, for some $\rho > 0$, and $\Phi_t(z) = z + it$, $t \geq 0$.*

The spectral value of an elliptic semigroup is clearly a holomorphic invariant.

The function h in the previous model is called the *Kœnigs function* of the semigroup. All the previous models are holomorphically non-equivalent. In fact, we have that
   – If $\Omega = \mathbb{C}$ and $\Phi_t(z) = e^{\lambda t}z$, for some $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda < 0$, then (ϕ_t) is a semigroup of *elliptic type*.
   – If $\Omega = \mathbb{C}$ and $\Phi_t(z) = z + it$, then (ϕ_t) is a semigroup of *parabolic type of zero hyperbolic step*.
   – If $\Omega = \mathbb{H}, \mathbb{H}^-$ and $\Phi_t(z) = z + it$, then (ϕ_t) is a semigroup of *parabolic type of positive hyperbolic step*.
   – If $\Omega = \mathbb{S}_\rho$, for some $\rho > 0$, and $\Phi_t(z) = z + it$, then (ϕ_t) is a semigroup of *hyperbolic type*.

In any of the non-elliptic cases, the model is given by $(\Omega = I \times \mathbb{R}, h, z + it)$, where I is any of the intervals $(-\infty, 0)$, $(0, +\infty)$, $(0, \rho)$, for some $\rho > 0$, or $\mathbb{R}$. This interval I is a holomorphic invariant. In particular, in the hyperbolic case, ρ depends on the dilation λ of the semigroup.

3.2. **Semigroups and topological models.** From a topological point of view, given a semigroup (ϕ_t), which is not a group, there are only two possible models:

Proposition 3.5. *Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$.*

   (1) *If (ϕ_t) is elliptic, not a group, then (ϕ_t) has a topological model given by $(\mathbb{C}, h, z \mapsto e^{-t}z)$.*

(2) *If (ϕ_t) is non-elliptic, then (ϕ_t) has a topological model of hyperbolic type, given by a model $(\mathbb{S}, h, z \mapsto z + it)$.*

Proof. Assume that (ϕ_t) is parabolic. Let $(\Omega, h, z \mapsto z + it)$ be the model of (ϕ_t) given by Theorem 3.4. If $\Omega = \mathbb{C}$, define $\varphi : \mathbb{C} \to \mathbb{S}$ by $\varphi(x + iy) = \theta(x) + iy$, where $\theta : \mathbb{R} \to (0, 1)$ is any homeomorphism. Then clearly $\varphi(z + it) = \varphi(z) + it$ for all $t \in \mathbb{R}$. Now, let $\widetilde{\varphi} := \varphi \circ h$. It is then easy to see that $(\mathbb{S}, \widetilde{\varphi}, \Phi_t)$ is a topological model for (ϕ_t). In case $\Omega = \mathbb{H}$ and $\Phi_t(z) = z + it$, it is enough to replace θ with any homeomorphism $\theta : (0, \infty) \to (0, 1)$. While, if $\Omega = \mathbb{H}^-$ one can replace θ with any homeomorphism $\theta : (-\infty, 0) \to (0, 1)$. If $\Omega = \mathbb{S}_\rho$, just define $\varphi(x + iy) = x/\rho + iy$.

In case (ϕ_t) is elliptic, not a group, let $(\mathbb{C}, h, z \mapsto e^{\lambda t} z)$ be the model of (ϕ_t) given by Theorem 3.4, with $\lambda = a + ib$, $a < 0$ and $b \in \mathbb{R}$. Define

$$\varphi(\rho e^{i\theta}) := \exp\left(-\left(\frac{1}{a} + i\frac{b}{a}\right)\log\rho\right)e^{i\theta}, \quad \rho \neq 0,$$

and $\varphi(0) = 0$. It is easy to see that $\varphi : \mathbb{C} \to \mathbb{C}$ is a homeomorphism and that $\varphi(e^{\lambda t} w) = e^{-t}\varphi(w)$ for all $t \in \mathbb{R}$ and $w \in \mathbb{C}$. Then $(\mathbb{C}, \widetilde{\varphi}, z \mapsto e^{-t} z)$ is the topological model of (ϕ_t), where $\tilde{\varphi} = \varphi \circ h$. $\square$

As a consequence we have the following straightforward result:

Corollary 3.6. *Every non-elliptic semigroup of holomorphic self-maps of $\mathbb{D}$ is topologically conjugated to a hyperbolic semigroup of holomorphic self-maps of $\mathbb{D}$.*

Clearly, elliptic and non-elliptic models cannot be topologically equivalent. For groups of automorphisms, there are more possible topologically inequivalent models:

Corollary 3.7. *Let (ϕ_t) be a group of automorphisms of $\mathbb{D}$.*
  (1) *If (ϕ_t) is elliptic with spectral value $i\theta$, with $\theta \in (-\pi, \pi]$, then its topological model is given by $(\mathbb{D}, h, z \mapsto e^{it|\theta|} z)$, with $h(\mathbb{D}) = \mathbb{D}$,*
  (2) *if (ϕ_t) is non-elliptic then its topological model is given by $(\mathbb{S}, h, z \mapsto z + it)$, with $h(\mathbb{D}) = \mathbb{S}$.*

Proof. If (ϕ_t) is non-elliptic, then the result follows at once from Proposition 3.5. If (ϕ_t) is elliptic, it has the holomorphic model $(\mathbb{D}, h, z \mapsto e^{it\theta} z)$ by Theorem 3.4. If $\theta < 0$, the map $z \mapsto \overline{z}$ conjugates the holomorphic model to the model $(\mathbb{D}, h, z \mapsto e^{it|\theta|} z)$ $\square$

Remark 3.8. Let (φ_t) and (ϕ_t) two groups of automorphisms with models $(\mathbb{D}, h, z \mapsto e^{it\theta_1} z)$ and $(\mathbb{D}, \tilde{h}, z \mapsto e^{it\theta_2} z)$, respectively. Then (φ_t) and (ϕ_t) are topologically conjugated if and only if $|\theta_1| = |\theta_2|$. Indeed, if $\theta_1 = -\theta_2$ the map $\tau(z) = \overline{z}$ intertwines the two groups. On the other hand, the semigroups are topologically conjugated, there exists a homeomorphism $T : \mathbb{D} \to \mathbb{D}$ such that $T(e^{\theta_1 t} z) = e^{\theta_2 t} T(z)$ for all $t > 0$ and all $z \in \mathbb{D}$. If $\theta_1 = 0$ then $\theta_2 = 0$ as well, and similarly if $\theta_2 = 0$ then $\theta_1 = 0$. In case θ_1, θ_2 are non-zero, let $t = 2\pi/\theta_1$. Then $T(z) = e^{\theta_2 2\pi/\theta_1} T(z)$ for all $z \in \mathbb{D}$. Hence $\theta_2/\theta_1 =: m \in \mathbb{Z}$. Taking $t = 2\pi/(m\theta_1)$ we deduce that $T(e^{2\pi/m} z) = T(z)$ for all $z \in \mathbb{D}$. Therefore $m = \pm 1$.

The previous remark follows also from Naishul's theorem (see, *e.g.*, [4, Theorem 2.29]), which states that two germs of elliptic holomorphic maps in $\mathbb{C}$ fixing 0 which are topologically conjugated by an orientation preserving map, must have the same derivative at 0. The proof of such a result is however much more complicated than the corresponding results for groups.

It is worth pointing out that while the absolute value of the spectral value is a topological invariant for elliptic groups, all elliptic semigroups which do not form a group, are topologically equivalent to an elliptic semigroup with spectral value -1.

4. MAXIMAL CONTACT ARCS AND BOUNDARY FIXED POINTS

4.1. **Trajectories.** The following result is due to P. Gumenyuk.

Theorem 4.1. [18, Thm. 3.1, Prop. 3.2] *(see also [10]). Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$. Then for every $\sigma \in \partial\mathbb{D}$ the non-tangential limit $\phi_t(\sigma) := \angle \lim_{z \to \sigma} \phi_t(z)$ exists. Moreover, the curve $[0, +\infty) \ni t \mapsto \phi_t(\sigma)$ is continuous.*

Given $\sigma \in \overline{\mathbb{D}}$, the curve $[0, +\infty) \ni t \mapsto \phi_t(\sigma)$ is called the *trajectory of the semigroup associated with σ*. If $\sigma \in \mathbb{D}$, the trajectory associated with σ is contained in $\mathbb{D}$ and converges to the Denjoy–Wolff point of the semigroup, in case (ϕ_t) is not an elliptic group.

In case $\sigma \in \partial\mathbb{D}$ and $\phi_t(\sigma) \in \mathbb{D}$ for some $t > 0$, then $\phi_s(\sigma) \in \mathbb{D}$ for all $s \geq t$. This motivates the following definition:

Definition 4.2. Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$ and $\sigma \in \partial\mathbb{D}$. The *life-time* (on the boundary) of the trajectory associated with σ is

$$T(\sigma) := \sup\{t \geq 0 : \phi_t(\sigma) \in \partial\mathbb{D}\} \in [0, +\infty].$$

Note that if σ is a boundary fixed point for a semigroup, then its associated trajectory is constant and the life-time $T(\sigma) = \infty$.

4.2. **Maximal contact arcs.** In this subsection we recall the notion of contact arc introduced by the first author and Gumenyuk in [6] and summarize some related results we will use throughout the paper.

Definition 4.3. Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$ with associated infinitesimal generator G. An open arc $A \subset \partial\mathbb{D}$ is said to be a *contact arc* for (ϕ_t), if G extends holomorphically to A with $\text{Re}\{\overline{z}\,G(z)\} = 0$ and $G(z) \neq 0$ for all $z \in A$. A contact arc A is *maximal* if there exists no other contact arc B for (ϕ_t) such that $B \supsetneq A$. Each contact arc A is endowed with a natural orientation induced by the unit vector field $G/|G|$. This determines the *initial* and *final* end-points of A.

If (ϕ_t) is a semigroup, not an elliptic group, and A is a contact arc for the semigroup, then $A \neq \partial\mathbb{D}$. Notice that for elliptic groups, $\partial\mathbb{D}$ is a contact arc. From now on, we assume that the semigroup is not an elliptic group.

The following results give a characterization of contact arcs in terms of the Kœnigs map.

Theorem 4.4. [6, Sect. 3]. *Let (ϕ_t) be a semigroup of holomorphic self-maps, not an elliptic group, with associated infinitesimal generator G and Kœnigs function h. Let $A \subset \partial\mathbb{D}$ be an open arc. Then the following statements are equivalent*

(1) *G extends holomorphically to A with $\mathsf{Re}\,\{\overline{z}\,G(z)\} = 0$ and $G(z) \neq 0$ for all $z \in A$;*

(2) *h extends holomorphically through A and*
  - *there exists $c \in \overline{I}$ such that $\mathsf{Re}\,h(z) = c$ for all $z \in A$, if (ϕ_t) is non-elliptic with a holomorphic model $(I \times \mathbb{R}, h, z + it)$;*
  - *$h(A)$ is contained in a spiral $\{z \in \mathbb{C} : z = e^{t\lambda}v, t \in \mathbb{R}\}$ for some $v \in \mathbb{C} \setminus \{0\}$, if (ϕ_t) is elliptic with spectral value λ.*

The relationship between contact arcs and trajectories is given by the following result. As a notation, if $A \subset \partial\mathbb{D}$ is an open sub-arc with end points p, q, we denote it by (p, q).

Theorem 4.5. [6, Sect. 3] *Let (ϕ_t) be a semigroup of holomorphic self-maps which is not an elliptic group.*

(1) *if A is a contact arc for (ϕ_t) and $x \in A$, then x is a regular contact point for ϕ_t for $t \in (0, T(x))$. Moreover, the map $t \mapsto \phi_t(x)$ is a homeomorphism of $(0, T(x))$ onto the open sub-arc $(x, \phi_{T(x)}(x)) \subseteq A$.*

(2) *If $p \in \partial\mathbb{D}$ is a contact point for ϕ_{t_0} for some $t_0 > 0$ and it is not a boundary fixed point, then there exists a maximal contact arc A for (ϕ_t) such that $p \in \overline{A}$.*

(3) *Let A be a maximal contact arc for (ϕ_t) with initial point x_0 and final point x_1. Then*
  (i) *either x_0 is a boundary fixed point of (ϕ_t) or x_0 is a contact point for ϕ_{t_0} for some $t_0 > 0$,*
  (ii) *either x_1 is the Denjoy–Wolff point of (ϕ_t) or $\phi_t(x_1) \in \mathbb{D}$ for all $t > 0$.*

Definition 4.6. Let (ϕ_t) be a semigroup of holomorphic self-maps which is not an elliptic group. A maximal contact arc for (ϕ_t) whose final point is the Denjoy–Wolff point of (ϕ_t) is called an *exceptional maximal contact arc*.

Clearly, a semigroup might have only two exceptional maximal contact arcs, and elliptic semigroups do not have any exceptional maximal contact arc. The life-time of a trajectory is continuous on each maximal contact arc:

Proposition 4.7. *Let (ϕ_t) be a semigroup of holomorphic self-maps which is not an elliptic group and A a maximal contact arc.*

(1) *if A is a exceptional, then $T(x) = \infty$ for all $x \in \overline{A}$.*

(2) *if A is not exceptional, then the function*
$$\overline{A} \ni x \mapsto T(x) \in [0, +\infty]$$

is continuous and strictly decreasing (with respect to the natural orientation of A induced by (ϕ_t)). Moreover, T is bounded from above on $\overline{A}$ if an only if the initial point of A is not a fixed point of (ϕ_t).

Proof. (1) follows immediately from Theorem 4.5.

(2) We prove the result in case (ϕ_t) is non-elliptic with holomorphic model $(I \times \mathbb{R}, h, z + it)$, the elliptic case follows similarly.

By Theorem 4.4, h extends holomorphically through A and therefore it follows that for all $x \in A$ and $t \geq 0$ it holds $h(\phi_t(x)) = h(x) + it$. Moreover, there exists $c \in \overline{I}$ such that $\mathsf{Re}\, h(z) = c$ for all $z \in A$. Let $R := \sup\{s \in \mathbb{R} : c + is \in h(A)\}$. Note that $R < \infty$, since A is not exceptional. Then

$$(4.1) \qquad\qquad\qquad T(x) = R - \mathsf{Im}\, h(x).$$

Now let $y \in A$. By Theorem 4.4, the map $t \mapsto \phi_t(y)$ is an homeomorphism of $(0, T(y))$ onto the open sub-arc $B := (y, \phi_{T(y)}(y))$ of A. Let $t(x)$ denote its inverse. Hence, for $x \in B$

$$T(x) = T(\phi_{t(x)}(y)) = R - \mathsf{Im}\, h(\phi_{t(x)}(y)) = R - \mathsf{Im}\, h(y) - t(x),$$

proving that T is continuous and strictly decreasing in B (with respect to the natural orientation of A induced by (ϕ_t)). By the arbitrariness of B, it follows that T is continuous and strictly decreasing in A.

If x_1 is the final point of A, since A is not exceptional, $T(x_1) = 0$, and the previous argument shows that T is continuous on $A \cup \{x_1\}$ as well.

Now we examine the initial point x_0. There are two cases. Either x_0 is a fixed point of (ϕ_t) or x_0 is a contact (not fixed) point of (ϕ_t). In the first case $T(x_0) = \infty$, while, in the second case, according to [6, Remark 3.4], $T(x_0) < \infty$. Let

$$S := \lim_{A \ni x \to x_0} T(x).$$

Since T is strictly decreasing, such a limit exists, finite or infinite.

Assume first that $T(x_0) = \infty$. By [6, Proposition 2.11(iii) and Lemma 2.10], $h(A) = \{c + is : s \in (-\infty, R)\}$. Hence, $S = \infty$ by (4.1).

Assume next $T(x_0) < \infty$. The curve $[0, \infty) \ni s \mapsto \phi_s(x_0)$ is continuous, and T is continuous on A. Hence, $\lim_{s \to 0} T(\phi_s(x_0)) = S$, and, clearly $S \leq T(x_0)$ by the definition of life time. On the other hand, if $S < T(x_0)$, let $t \in (S, T(x_0))$. Taking into account that T is strictly decreasing on A, it follows that $\phi_t(\phi_s(x_0)) \in \mathbb{D}$ for all $s > 0$, hence $\phi_t(x_0) \in \mathbb{D}$, a contradiction. □

For our aims, we need to recall from [6] how the maximal contact arcs are related to the Kœnigs function of the hyperbolic and starlike elliptic models:

Proposition 4.8. [6, Sect. 3] *Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$. Let (Ω, h, Φ_t) be a holomorphic model of (ϕ_t).*

(1) *If (ϕ_t) is a starlike elliptic semigroup, not a group, then the initial point of a maximal contact arc A is a fixed point if and only if the image of $h(A)$ is not bounded.*

(2) *If (ϕ_t) is a non-elliptic semigroup, the initial point of a maximal contact arc A is a fixed point if and only if $\operatorname{Im} h(A)$ is not bounded from below. Moreover, if $\Omega = I \times \mathbb{R}$, then A is exceptional if and only if either $I = (c, a)$ with $c \in \mathbb{R}, a \in (c, +\infty]$ or $I = (a, c)$ with $c \in \mathbb{R}$ and $a \in [-\infty, c)$, and there exists $d \in [-\infty, +\infty)$ such that $\{c + it : t > d\} = h(A)$.*

4.3. **Boundary fixed points.** Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$. Assume (ϕ_t) is either non-elliptic with holomorphic model $(\Omega, h, z \mapsto z + it)$ or starlike elliptic with the model $(\mathbb{C}, h, z \mapsto e^{-t}z)$. Let $Q := h(\mathbb{D})$. Set

$$(4.2) \qquad A(\phi_t) := \begin{cases} \cap_{t \geq 0}(e^{-t}Q) & \text{if } (\phi_t) \text{ is starlike elliptic,} \\ \cap_{t \geq 0}(Q + it) & \text{if } (\phi_t) \text{ is non-elliptic.} \end{cases}$$

Following ideas from [8, Thm. 2.5], it is possible to prove that, given a non-elliptic semigroup, if for some $c \in \mathbb{R}$ the line $\{\zeta \in \mathbb{C} : \operatorname{Re} \zeta = c\}$ is contained in $A(\phi_t)$ then $p := \lim_{t \to -\infty} h^{-1}(c + it)$ is a boundary fixed point of the semigroup. Next result shows a characterization of those boundary fixed points that can be reached in this way.

Proposition 4.9. *Let (ϕ_t) be a non-elliptic semigroup of holomorphic self-maps of $\mathbb{D}$. Let C be a connected components of $A(\phi_t)$. Then $C = C + it$, for all $t > 0$, and there exist $-\infty \leq a \leq b \leq +\infty$ such that $\overline{C} = \{\zeta \in \mathbb{C} : a \leq \operatorname{Re} \zeta \leq b\}$.*

Moreover, there is a one-to-one correspondence between boundary regular fixed points of (ϕ_t), different from the Denjoy–Wolff point, and connected components of $A(\phi_t)$ for which there exist $-\infty < a < b < +\infty$ such that $\overline{C} = \{\zeta \in \mathbb{C} : a \leq \operatorname{Re} \zeta \leq b\}$ associating to C the point $p(C) := \lim_{t \to -\infty} h^{-1}(c + it)$, where c is any point in (a, b), and $p(C) \in \partial\mathbb{D}$ is a boundary regular fixed point of (ϕ_t).

Also, there is a one-to-one correspondence between boundary super-repelling fixed points of (ϕ_t) which are not inital points of a maximal contact arc of (ϕ_t) and connected components of $A(\phi_t)$ given by a line associating to the connected component $C = \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta = c\}$ the point $p(C) := \lim_{t \to -\infty} h^{-1}(c + it)$, and $p(C) \in \partial\mathbb{D}$ is a boundary super-repelling fixed point of (ϕ_t).

Proof. The case of boundary regular fixed points is in [8, Thm. 2.5].

By [6, Lemma 4.3], if p is a super-repelling boundary fixed point of (ϕ_t) for which there is no connected component of $A(\phi_t)$ of the form $C = \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta = c\}$ with $p := \lim_{t \to -\infty} h^{-1}(c + it)$ then p is the initial point of a maximal contact arc. Conversely, assume by contradiction that p is a boundary super-repelling fixed point which is the initial point of a maximal contact arc A and there exists $c \in \mathbb{R}$ such that $p := \lim_{t \to -\infty} h^{-1}(c + it)$.

Take a sequence (t_n) converging to $-\infty$ such that the connected component of $h(\mathbb{D}) \cup (\mathbb{R} + it_n)$ containing $c + it_n$ is of the form $(c - \alpha_n, c + \beta_n)$, where $0 \leq \alpha_{n+1} < \alpha_n$ and

$0 \leq \beta_{n+1} < \beta_n$ for all $n \in \mathbb{N}$. Since p is super-repelling, the sequences (α_n) and (β_n) converge to zero. Consider the Jordan arc $C_n = h^{-1}((c - \alpha_n, c + \beta_n))$. Since h has no Koebe arcs, the diameter of (C_n) tends to zero. Therefore, for n large enough, one of the end points of the Jordan arc C_n belongs to A, say this end point is $\lim_{C_n \ni z \to c+\beta_n} h^{-1}(z)$. By Theorem 4.4(2), $\mathsf{Re}\, h(z) = c + \beta_n = c + \beta_{n+1}$ for all $z \in A$, contradicting $\beta_{n+1} < \beta_n$. $\square$

A similar statement holds for the starlike elliptic case, replacing open strips $\{z \in \mathbb{C} : a \leq \mathsf{Re}\, z \leq b\}$ with angles $\{z \in \mathbb{C} : a \leq \arg z \leq b\}$ and vertical lines with half-lines $\{z = tv : t \geq 0\}$, $v \in \mathbb{C} \setminus \{0\}$.

Definition 4.10. Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$. A boundary super-repelling fixed point $p \in \partial\mathbb{D}$ for (ϕ_t) is called

(1) a boundary super-repelling fixed point *of first type* if it corresponds to a connected component of $A(\phi_t)$ which is a line,

(2) a boundary super-repelling fixed point *of second type* if it is the initial point of a maximal contact arc for (ϕ_t) which is not exceptional.

(3) a boundary super-repelling fixed point *of third type* if it is the initial point of an exceptional maximal contact arc for (ϕ_t).

Note that in case (1) there exists a backward orbit of (ϕ_t) which lands at p. Indeed, if p corresponds to the connected component $C = \{\zeta \in \mathbb{C} : \mathsf{Re}\, \zeta = c\}$ of $A(\phi_t)$, then the curve $[0, +\infty] \ni t \mapsto h^{-1}(c + it) \in \mathbb{D}$ is a backward orbit of (ϕ_t) landing at p.

By [6, Lemma 4.3], a super-repelling fixed point is necessarily one of the above types. In particular, we have the following result:

Corollary 4.11. *Let (ϕ_t) be a semigroup of holomorphic self-maps of $\mathbb{D}$. A boundary super-repelling fixed point $p \in \partial\mathbb{D}$ for (ϕ_t) is either the starting point of a maximal contact arc or there exists a backward orbit of (ϕ_t) landing at p.*

Remark 4.12. If (ϕ_t) is elliptic, then there are no boundary super-repelling fixed points of third type, while, if (ϕ_t) is non-elliptic then there are at most two boundary super-repelling fixed points of third type.

Next result is due to P. Gumenyuk and shows the relationship between the boundary behavior of Kœnigs functions and boundary fixed points.

Theorem 4.13. [18, Prop. 3.4, Prop. 3.7] *Let (ϕ_t) be a non-elliptic semigroup of holomorphic self-maps of $\mathbb{D}$ with holomorphic model $(\Omega, h, z \mapsto z + it)$. Then*

(1) *for every $\sigma \in \partial\mathbb{D}$ the non-tangential limit $\angle \lim_{z \to \sigma} h(z) \in \overline{\mathbb{C}}$ exists.*

(2) *Let $\sigma \in \partial\mathbb{D}$. The (unrestricted) limit $\lim_{z \to \sigma} \mathsf{Re}\, h(z)$ exists finitely if and only if σ is not a regular fixed point for (ϕ_t).*

(3) *$\sigma \in \partial\mathbb{D}$ is a boundary fixed point for (ϕ_t) different from the Denjoy–Wolff point if and only if $\lim_{z \to \sigma} \mathsf{Im}\, h(z) = -\infty$.*

(4) *If $\sigma \in \partial\mathbb{D}$ is not the Denjoy–Wolff of (ϕ_t) then $\limsup_{z\to\sigma} \operatorname{Im} h(z) < +\infty$.*

A similar result (see [18]) holds for the starlike elliptic case replacing the real and imaginary part of h with the modulus and the argument of h.

In the sequel we will need also the following characterization of points belonging to exceptional maximal contact arcs in terms of Carathéodory's prime end theory (see, *e.g.* [7, Chapter 9], [20, Sections 2.4-2.5]). As a matter of notation, if $h : \mathbb{D} \to \mathbb{C}$ is univalent, we denote by $\hat{h}$ the homeomorphism induced by h from $\partial\mathbb{D}$ and the space of Carathéodory's prime ends of $h(\mathbb{D})$.

Proposition 4.14. *Let (ϕ_t) be a non-elliptic semigroup of holomorphic self-maps of $\mathbb{D}$. Let $(I \times \mathbb{R}, h, z \mapsto z+it)$ be the holomorphic model of (ϕ_t), where $I = \mathbb{R}, (0, +\infty), (-\infty, 0)$ or $(0, \rho)$ for some $\rho > 0$. Let M be the union of the exceptional maximal contact arcs for (ϕ_t) (possibly $M = \emptyset$). Let $p \in \partial\mathbb{D}$ be different from the Denjoy–Wolff point of (ϕ_t). Then $p \notin \overline{M}$ if and only if the following condition holds:*

(L) *there exists a null chain (C_n) in $h(\mathbb{D})$ representing $\hat{h}(p)$ such that there exist a compact subinterval $I' \subset I$ and $N \in \mathbb{N}$ such that the connected component V_n of $h(\mathbb{D}) \setminus C_n$ which does not contain C_0 satisfies $V_n \subset I' \times \mathbb{R}$ for all $n \geq N$.*

Proof. Assume $\hat{M}$ is an exceptional maximal contact arc for (ϕ_t). By Proposition 4.8, $I \neq \mathbb{R}$ and we can assume without loss of generality that $I = (0, \rho)$, with $\rho \in (0, +\infty]$ and that $h(\hat{M}) = \{it, t > d\}$ for some $d \in [-\infty, +\infty)$.

Let $p \in \partial\mathbb{D}$ be a point different from the Denjoy–Wolff point of (ϕ_t).

If $p \in \hat{M}$, since by Theorem 4.4, h extends holomorphically through p and $h(p) \in \{it, t > d\}$, every null chain (C_n) representing $\hat{h}(p)$ converges to $\{h(p)\}$, hence it is not possible to find a closed subinterval $I' \subset I$ and $N \in \mathbb{N}$ such that $V_n \subset I' \times \mathbb{R}$ for all $n \geq N$.

If p is the initial point of $\hat{M}$, assume by contradiction that (C_n) is a null chain representing $\hat{h}(p)$ such that there exists a closed subinterval $I' = [a, b] \subset I$, for $0 < a < b$, such that $V_n \subset I' \times \mathbb{R}$ for n sufficiently big. By Theorem 4.13, h has non-tangential limit (finite or infinite) at p. Also $\lim_{\hat{M} \ni z \to p} h(z)$ exists (finite or infinite). Indeed, $\operatorname{Re} h(z) = 0$ and $\operatorname{Im} h(\phi_t(z)) = \operatorname{Im} h(z) + t$ for all $z \in \hat{M}$ and all $t \geq 0$, proving that $\operatorname{Im} h(z)$ decreases when z moves on $\hat{M}$ toward the initial point p of $\hat{M}$. By [6, Lemma 2.10.(1)], $\lim_{\hat{M} \ni z \to p} h(z) = \angle\lim_{z \to p} h(z)$. In particular, $\angle\lim_{r \to 1^-} \operatorname{Re} h(rp) = 0$. Since for every $n \in \mathbb{N}$ the curve $(0, 1) \ni r \mapsto h(rp)$ is eventually contained in V_n, and since $V_n \subset \{\operatorname{Re} z > a\}$, it follows that $\lim_{r \to 1^-} \operatorname{Re} h(rp) \geq a > 0$, a contradiction. Therefore (L) implies $p \notin \hat{\overline{M}}$.

Now assume $p \notin \overline{M}$.

Suppose first p is a boundary regular fixed point of (ϕ_t). By Proposition 4.9, there exist $-\infty < a < b < +\infty$ such that $(a, b) \subset I$, $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\} \subset h(\mathbb{D})$ but $\{z : a - \epsilon < \operatorname{Re} z < b + \epsilon\} \not\subset h(\mathbb{D})$ for any $\epsilon > 0$, and for every $c \in (a, b)$, $\lim_{t \to -\infty} h(c + it) = p$.

Let C_n be the connected component of $\{\operatorname{Im} z = -n\}$ which intersects $\{z : a < \operatorname{Re} z < b\}$. Then (C_n) is a null chain which represents $\hat{h}(p)$. If $I = \mathbb{R}$ then (C_n) satisfies (L) and we are done. Assume $I = (\alpha, \beta)$, with $-\infty < \alpha \leq a < b \leq \beta \leq +\infty$. If $\alpha = a$, then clearly p is the initial point of the exceptional maximal contact arc defined by $\{z \in \partial\mathbb{D} : z = h^{-1}(a + it), t \in \mathbb{R}\}$, therefore $\alpha < a$. Similarly $b < \beta$. Hence (C_n) satisfies condition (L).

Now assume p is not a boundary regular fixed point and suppose by contradiction that condition (L) is not satisfied. Let (C_n) be a null chain and the V_n's as defined above. The impression of $\hat{h}(p)$ is given by $I(\hat{h}(p)) = \cap_{n \in \mathbb{N}} \overline{V_n}$ (where the closure has to be understood in the Riemann sphere $\mathbb{CP}^1$). It is known that $x \in I(\hat{h}(p))$ if and only if there exists a sequence $\{z_n\} \subset \mathbb{D}$ such that $z_n \to p$ and $h(z_n) \to x$.

By Theorem 4.13.(2), there exists $a \in \mathbb{R}$ such that $\lim_{z \to p} \operatorname{Re} h(z) = a$. Therefore, if (L) does not hold, it follows that $a \in \partial I$. In particular, if $I = \mathbb{R}$, we obtain a contradiction and hence (L) holds in this case. Thus we can assume without loss of generality that $I = (a, b)$, with $-\infty < a < b \leq +\infty$.

Since $\limsup_{z \to p} \operatorname{Im} h(z) < +\infty$ by Theorem 4.13.(4), the impression of $\hat{h}(p)$ is given by

$$I(\hat{h}(p)) = \{z : \operatorname{Re} z = a, \operatorname{Im} z \in [k_1, k_2]\},$$

for some $-\infty \leq k_1 < k_2 < +\infty$ or $I(\hat{h}(p))$ is the point at infinity in $\mathbb{CP}^1$. Let $\Gamma = h([0,1)p)$. Such a curve has a limit $q \in \mathbb{CP}^1$ by Theorem 4.13.(1). Moreover, since $\lim_{z \to p} \operatorname{Re} h(z) = a$, it follows that $\overline{\Gamma}$ intersects the line $\{\operatorname{Re} z = a\}$ at q. In case q is the point at infinity, let us set $\operatorname{Im} q = -\infty$. With such a notation, since $\Gamma + it \subset h(\mathbb{D})$ for all $t \geq 0$, it follows that $\{z : \operatorname{Re} z = a, \operatorname{Im} z > \operatorname{Im} q\} \subset \partial h(\mathbb{D})$. Therefore (ϕ_t) has an exceptional maximal contact arc $\hat{M}$ given by $h^{-1}(\{(a + it) : t > \operatorname{Im} q\})$. We claim that $p \in \overline{\hat{M}}$, reaching a contradiction. This follows at once from

(4.3) $$\lim_{t \to (\operatorname{Im} q)^+} h^{-1}(a + it) = p.$$

In order to prove (4.3) , let $D(q, \frac{1}{n})$ be a disc in the spherical metric centered at q and with radius $1/n$, $n \in \mathbb{N}$. Note that for all $n \in \mathbb{N}$, $\Gamma \cap D(q, \frac{1}{n}) \neq \emptyset$. Therefore, for each $n \in \mathbb{N}$ we can find a Jordan curve $T_n : [0,1] \to \mathbb{C}$ such that $T_n([0,1]) \subset D(q, \frac{1}{n})$, $T_n((0,1)) \subset h(\mathbb{D})$, $T_n(0) = a + it_n$ for some $t_n > \operatorname{Im} q$, $T_n(1) \in \Gamma$. Let $Q_n := T_n(0,1)$. Then, the spherical diameters of Q_n tend to 0. Moreover, $h^{-1}(Q_n)$ is a sequence of Jordan arcs in $\mathbb{D}$, whose Euclidean diameter tends to 0 by the no Koebe arcs theorem (see, *e.g.*, [19, Corollary 9.1]). Hence (4.3) holds. $\square$

5. Extension of topological conjugation for non-elliptic semigroups

Lemma 5.1. *Let (ϕ_t) and (φ_t) be two non-elliptic semigroups in $\mathbb{D}$. Let $(\Omega_1, h_1, z \mapsto z + ti)$ be the holomorphic model of (ϕ_t) and let $(\Omega_2, h_2, z \mapsto z + ti)$ be the holomorphic model of (φ_t). Write $\Omega_j = I_j \times \mathbb{R}$, $j = 1, 2$, where I_j is any of the intervals $(-\infty, 0)$, $(0, +\infty)$,*

$(0, \rho)$, *with $\rho > 0$, or $\mathbb{R}$. Then (ϕ_t) and (φ_t) are topologically conjugated if and only if there exist an homeomorphism $u : I_1 \to I_2$ and a continuous function $v : I_1 \to \mathbb{R}$ such that*

(5.1) $$\tau(z) := u(\operatorname{Re} z) + i(\operatorname{Im} z + v(\operatorname{Re} z))$$

satisfies $\tau(h_1(\mathbb{D})) = h_2(\mathbb{D})$.

Proof. Let $Q_1 := h_1(\mathbb{D})$ and let $Q_2 := h_2(\mathbb{D})$. By Proposition 3.3, (ϕ_t) and (φ_t) are topologically conjugated if and only if there exists a homeomorphism $\tau : \Omega_1 \to \Omega_2$ such that $\tau(z + it) = \tau(z) + it$ for all $t \in \mathbb{R}$ and $\tau(Q_1) = Q_2$.

Write $z = x + iy$ with $x \in I_1$ and $y \in \mathbb{R}$. Then $\tau(x + iy) = \tau(x) + iy$. Write $\tau(x) = u(x) + iv(x)$. Then $u : I_1 \to I_2$ and $v : I_1 \to \mathbb{R}$ are continuous. Since also $\tau^{-1} : \Omega_2 \to \Omega_1$ satisfies $\tau^{-1}(z + it) = \tau^{-1}(z) + it$ for all $t \in \mathbb{R}$, it follows that u is a homeomorphism. $\square$

Let (φ_t) be a non-elliptic semigroup, let us denote by $M(\varphi_t) \subset \partial\mathbb{D}$ the set of points which belong to an exceptional maximal contact arc for (φ_t) (note that $M(\varphi_t)$ is open and possibly empty). Let $x \in \partial\mathbb{D}$ be the Denjoy–Wolff point of (φ_t) and let

$$E(\varphi_t) := \overline{M(\varphi_t) \cup \{x\}}.$$

Proposition 5.2. *Let (ϕ_t) and (φ_t) be two non-elliptic semigroups of holomorphic self-maps of $\mathbb{D}$. Suppose (ϕ_t) and (φ_t) are topologically conjugated via the homeomorphism $f : \mathbb{D} \to \mathbb{D}$. Then f extends to a homeomorphism*

$$f : \overline{\mathbb{D}} \setminus E(\phi_t) \to \overline{\mathbb{D}} \setminus E(\varphi_t).$$

Moreover, for all $p \in \overline{\mathbb{D}} \setminus E(\phi_t)$ it holds $T(p) = T(f(p))$ and $f(\phi_t(p)) = \varphi_t(f(p))$ for all $t \geq 0$.

Proof. In order to prove the result, we use Carathéodory's prime ends theory (see, *e.g.* [7, Chapter 9], [20, Sections 2.4-2.5]). Let $(\Omega_1 = I_1 \times \mathbb{R}, h_1, z \mapsto z + it)$ be the holomorphic model of (ϕ_t) and let $(\Omega_2 = I_2 \times \mathbb{R}, h_2, z \mapsto z + it)$ be the holomorphic model of (φ_t), where I_1, I_2 are intervals of the form $(-\infty, 0)$, $(0, +\infty)$, $(0, \rho)$, with $\rho > 0$, or $\mathbb{R}$. Let $\hat{h}_1$ denote the homeomorphism from $\partial\mathbb{D}$ to the Carathéodory prime-ends boundary of $h_1(\mathbb{D})$ defined by h_1.

Let $p \in \partial\mathbb{D} \setminus E(\phi_t)$. Let (C_n) be a null-chain in $h_1(\mathbb{D})$ which represents $\hat{h}_1(p)$ and denote by V_n the connected component of $h_1(\mathbb{D}) \setminus C_n$ which does not contain C_0. Let $I(\hat{h}_1(p)) = \cap_{n>0} \overline{V_n}$ denote the impression of $\hat{h}_1(p)$ in the Riemann sphere (here $\overline{V_n}$ denotes the closure of V_n in the Riemann sphere). Then, for any sequence $\{z_n\} \subset \mathbb{D}$ converging to p, the sequence $\{h(z_n)\}$ is eventually contained in V_m for all $m \geq 0$. Conversely, if $\{w_n\} \subset h_1(\mathbb{D})$ is any sequence which is eventually contained in V_m for any $m \in \mathbb{N}$, then $\lim_{n\to\infty} h_1^{-1}(w_n) = p$.

Since $\limsup_{z \to p} \mathsf{Im}\, h_1(z) < K$ for some constant $K < +\infty$ by Theorem 4.13.(4), it follows that

$$(5.2) \qquad\qquad \sup\{\mathsf{Im}\, z : z \in I(\hat{h}_1(p))\} < +\infty.$$

By Proposition 4.14, we can assume that $V_n \subset I_1' \times \mathbb{R}$ for some compact subinterval $I' \subset I$ and for all $n \in \mathbb{N}$. By Lemma 5.1, $f = h_2^{-1} \circ \tau \circ h_1$, where τ is given by (5.1). In particular, setting $\tau(\infty) := \infty$, it follows that τ is uniformly continuous on $\overline{I_1' \times \mathbb{R}}$ (where the closure has to be understood in $\mathbb{CP}^1$ and ∞ denotes the point at infinity). From this it follows easily that $\tau(C_n)$ is a null chain in $h_2(\mathbb{D})$. Moreover, if W_n is the connected component of $h_2(\mathbb{D}) \setminus \tau(C_n)$ which does not contain $\tau(C_0)$, then $W_n = \tau(V_n)$ and W_n is contained in $u(I_1') \times \mathbb{R}$ for all $n \in \mathbb{N}$, with $u(I_1')$ being a compact subinterval of I_2. Let $q \in \partial\mathbb{D}$ be such that $(\tau(C_n))$ represents $\hat{h}_2(q)$. Then $\tau(I(\hat{h}_1(p))) = I(\hat{h}_2(q))$, and by (5.2),

$$\sup\{\mathsf{Im}\, z : z \in I(\hat{h}_2(q))\} < +\infty,$$

which, in turns, implies that $\limsup_{z \to q} \mathsf{Im}\, h_2(z) < +\infty$. Since the non-tangential limit of $\mathsf{Im}\, h_2$ at the Denjoy–Wolff point of (φ_t) is $+\infty$, this implies that q is not the Denjoy–Wolff point of (φ_t). Hence, by Proposition 4.14, q does not belong to the closure of an exceptional maximal compact arc for (φ_t).

Now we show that, in fact, $\lim_{z \to p} f(z) = q$. Let $\{z_n\}$ be a sequence converging to p. Then $\{h_1(z_n)\}$ is eventually contained in V_m for all $m \in \mathbb{N}$. Thus $\{\tau(h_1(z_n))\}$ is eventually contained in $\tau(V_m) = W_m$ for all $m \in \mathbb{N}$, which implies that $\{h_2^{-1}(\tau(h_1(z_n)))\}$ converges to q, that is, $\lim_{z \to p} f(z) = q$.

Next, if $p_1, p_2 \in \partial\mathbb{D} \setminus E(\phi_t)$ are two different points, the null chains $(C_n^1), (C_n^2)$ in $h_1(\mathbb{D})$ which represents $\hat{h}_1(p_1)$ and $\hat{h}_1(p_2)$ are not equivalent (in the prime-ends sense). Therefore, chosen those chains to satisfy condition (L) in Lemma 4.14 for all $n \in \mathbb{N}$, it follows easily that $(\tau(C_n^1))$ and $(\tau(C_n^2))$ are not equivalent. Thus, if $q_j \in \partial\mathbb{D}$ is such that $(\tau(C_n^j))$ represents $\hat{h}_2(q_j)$, $j = 1, 2$, it follows that $q_1 \neq q_2$. Hence, the extension of f to $\partial\mathbb{D} \setminus E(\phi_t)$ is injective.

Finally, we prove that $f : \overline{\mathbb{D}} \setminus E(\phi_t) \to \overline{\mathbb{D}} \setminus E(\varphi_t)$ is continuous.

Recall the following fact from Carathéodory's prime ends theory. Let $h : \mathbb{D} \to h(\mathbb{D})$ be univalent. Let $\{p_m\}$ be a sequence of points of $\partial\mathbb{D}$. For each $m \in \mathbb{N}$, let (C_n^m) be a null chain in $h(\mathbb{D})$ representing the prime end $\hat{h}(p_n)$. Then $p_m \to p \in \partial\mathbb{D}$ if and only if, given (C_n) a null chain in $h(\mathbb{D})$ representing $\hat{h}(p)$, for every open subset $U \subset h(\mathbb{D})$ such that $C_n \subset U$ for all $n \gg 1$, it follows that there exists $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ there exists $N_m \in \mathbb{N}$ such that $C_n^m \subset U$ for all $m \geq m_0$ and $n \geq N_m$.

In order to prove continuity of f, it is enough to show that if $\{p_m\} \subset \partial\mathbb{D} \setminus E(\phi_t)$ converges to $p \in \partial\mathbb{D} \setminus E(\phi_t)$ then $\lim_{m \to \infty} f(p_m) = f(p)$. Let $q_m := f(p_m), q := f(p)$.

For each $m \in \mathbb{N}$, let (C_n^m) be a null chain in $h_1(\mathbb{D})$ representing $\hat{h}_1(p_m)$, chosen so that condition (L) of Proposition 4.14 is satisfied for all n, and similarly let (C_n) be a null chain which satisfies condition (L) and represents $\hat{h}_1(p)$.

As shown before, $(\tau(C_n^m))$ is a null chain in $h_2(\mathbb{D})$ representing $\hat{h}_2(q_m)$ and $(\tau(C_n))$ is a null chain in $h_2(\mathbb{D})$ representing $\hat{h}_2(q)$.

Fix an open subset $U \subset h_2(\mathbb{D})$ such that $\tau(C_n) \subset U$ for $n >> 1$. Then, $\tau^{-1}(U)$ eventually contains (C_n). Since $p_m \to p$, this implies that for every $m >> 1$, (C_n^m) is eventually contained in $\tau^{-1}(U)$. Hence $(\tau(C_n^m))$ is eventually contained in U for $m >> 1$. Therefore, $q_m \to q$ and f is continuous.

Since the same applies to f^{-1}, it follows that $f : \overline{\mathbb{D}} \setminus E(\phi_t) \to \overline{\mathbb{D}} \setminus E(\varphi_t)$ is a homeomorphism.

In order to prove the last equations, fix $p \in \overline{\mathbb{D}} \setminus E(\phi_t)$ and fix $t \geq 0$. Let $r \in (0, 1)$. Then $\varphi_t(f(rp)) = f(\phi_t(rp)) \to f(\phi_t(p))$ as $r \to 1$. Therefore, the limit of φ_t along the continuous curve $r \mapsto f(rp)$ (which converges to $f(p)$) is $f(\phi_t(p))$. By the Lindelöf theorem, $\varphi_t(f(p)) = \angle \lim_{z \to f(p)} \varphi_t(z) = f(\phi_t(p))$, and thus the functional equation holds at p. From this, it follows at once that $T(p) = T(f(p))$. $\qquad\square$

6. Topological invariants for non elliptic semigroups

First we examine maximal contact arcs which are not exceptional:

Proposition 6.1. *Let (ϕ_t) and (φ_t) be two non-elliptic semigroups in $\mathbb{D}$. Suppose (ϕ_t) and (φ_t) are topologically conjugated via the homeomorphism $f : \mathbb{D} \to \mathbb{D}$. If W is a maximal contact arc for (ϕ_t) which is not exceptional, then f extends to a homeomorphism from W onto $f(W)$ and $f(W)$ is a maximal contact arc for (φ_t) which is not exceptional.*

Proof. Since W is not exceptional, then $W \subset \partial\mathbb{D} \setminus E(\phi_t)$. Thus, by Proposition 5.2, f extends as a homeomorphism on W and $f(W) \subset \partial\mathbb{D} \setminus E(\varphi_t)$. Therefore, we are left to show that $f(W)$ is a maximal contact arc.

Let $p \in W$. By Proposition 5.2, $T(p) = T(f(p)) > 0$, hence $\varphi_s(f(p)) = f(\phi_s(p))$, $s \in (0, T(p))$, belong to a maximal contact arc $W' \subset f(W)$ for (φ_t). If $q \in f(W) \setminus W'$, let $x_0 \in W$ be such that $f^{-1}(q) = \phi_s(x_0)$, for some $s > 0$. But then, again by Proposition 5.2, $f(\phi_s(x_0))$, $s \in (0, T(x_0))$ is a contact arc which contains W', contradicting its maximality. Thus $f(W)$ is a maximal contact arc. $\qquad\square$

Now we consider boundary fixed points and their type. Again, the result is an application of Proposition 5.2.

Proposition 6.2. *Let (ϕ_t) and (φ_t) be two non-elliptic semigroups in $\mathbb{D}$. Suppose (ϕ_t) and (φ_t) are topologically conjugated via the homeomorphism $f : \mathbb{D} \to \mathbb{D}$. Let $p \in \partial\mathbb{D}$ be a boundary fixed point for (ϕ_t). Suppose that $p \notin E(\phi_t)$. Then the unrestricted limit*

$$f(p) := \lim_{z \to p} f(z) \in \partial\mathbb{D}$$

exists and $f(p) \in \partial\mathbb{D} \setminus E(\varphi_t)$ is a boundary fixed point for (φ_t). Moreover,

    (1) *if p is a boundary regular fixed point for (ϕ_t) then $f(p)$ is a boundary regular fixed point for (φ_t),*

(2) *if p is a boundary super-repelling fixed point of first type (respectively of second type) for (ϕ_t) then $f(p)$ is a a boundary super-repelling fixed point of first type (respectively of second type) for (φ_t),*

Proof. First assume p is a boundary fixed point which does not start an exceptional maximal contact arc. By Proposition 5.2, f extends continuously at p, $f(p)$ is not in the closure of an exceptional maximal contact arc and $T(p) = T(f(p))$. Therefore, $f(p)$ is a boundary fixed point for (φ_t).

In order to show that the type of p is preserved let $(\Omega_1 = I_1 \times \mathbb{R}, h_1, z \mapsto z + ti)$ be the holomorphic model of (ϕ_t) and let $(\Omega_2 = I_2 \times \mathbb{R}, h_2, z \mapsto z + ti)$ be the holomorphic model of (φ_t). Let $Q_1 := h_1(\mathbb{D})$ and let $Q_2 := h_2(\mathbb{D})$. By Lemma 5.1, (ϕ_t) and (φ_t) are topologically conjugated if and only if there exists a homeomorphism $\tau : \Omega_1 \to \Omega_2$ given by (5.1) and $\tau(Q_1) = Q_2$. By Proposition 3.3, $f = h_2^{-1} \circ \tau \circ h_1$.

Then the restriction of τ to $A(\phi_t)$ induces a homeomorphism from $A := A(\phi_t)$ to $\tilde{A} := A(\varphi_t)$. Indeed,

$$\tilde{A} = \cap_{t \geq 0}(Q_2 + it) = \cap_{t \geq 0}(\tau(Q_1) + it)$$
$$= \cap_{t \geq 0}\tau(Q_1 + it) = \tau\left(\cap_{t \geq 0}(Q_1 + it)\right) = \tau(A).$$

In particular, every connected component of $\tilde{A}$ corresponds to exactly one connected component of A.

Since τ defines a homeomorphism between A and $\tilde{A}$ and maps vertical lines $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta = c\}$ to vertical lines, it follows that to each connected component of A with non-empty interior (respectively with empty interior) there corresponds exactly one connected component of $\tilde{A}$ with non-empty interior (resp. with empty interior).

From this, it follows immediately that if p is a boundary regular fixed point (respectively a super-repelling fixed point of first type), then $f(p)$ is a boundary regular fixed point (resp. a super-repelling fixed point of first type). Moreover, if p is a super-repelling fixed point of second type, then so is $f(p)$, for otherwise one can apply the previous conclusion to f^{-1} at $f(p)$ to get a contradiction. Hence, (1) and (2) are proved. □

Another trivial consequence of Proposition 5.2 is that the continuity of a semigroup at a boundary point is a topological invariant:

Proposition 6.3. *Let (ϕ_t) and (φ_t) be two non-elliptic semigroups in $\mathbb{D}$. Suppose (ϕ_t) and (φ_t) are topologically conjugated via the homeomorphism $f : \mathbb{D} \to \mathbb{D}$. Let $p \in \partial\mathbb{D} \setminus E(\phi_t)$ be such that the unrestricted limit $\lim_{z \to p} \phi_t(z)$ exists for all $t \geq 0$. Then the unrestricted limit $f(p) = \lim_{z \to p} f(z) \in \partial\mathbb{D} \setminus E(\varphi_t)$ and the unrestricted limit $\lim_{z \to f(p)} \varphi_t(z)$ exist for all $t \geq 0$.*

Proof. By Proposition 5.2, f has unrestricted limit at p, $f(p) \notin E(\varphi_t)$ and f^{-1} extends continuously at $f(p)$, $f^{-1}(f(p)) = p$. Since $\varphi_t(z) = f(\phi_t(f^{-1}(z)))$, $z \in \mathbb{D}$, the result follows. □

7. Exceptional maximal contact arcs and Denjoy – Wolff point

In this section we examine the behavior of the topological conjugation on exceptional maximal contact arcs and boundary Denjoy – Wolff points. The behavior of the topological intertwining map on an exceptional maximal contact arc can be quite wild.

We start with the following result:

Proposition 7.1. *Let (ϕ_t) and (φ_t) be two non-elliptic semigroups in $\mathbb{D}$. Suppose (ϕ_t) and (φ_t) are topologically conjugated via the homeomorphism $f : \mathbb{D} \to \mathbb{D}$.*

(1) *If $p \in \partial\mathbb{D}$ is a boundary regular fixed point for (ϕ_t) which starts an exceptional maximal contact arc, then the non-tangential limit $f(p) = \angle \lim_{z \to p} f(z)$ exists and $f(p)$ is a boundary regular fixed point for (φ_t) (possibly the Denjoy – Wolff point of (φ_t)). In fact, $f(p) \in E(\varphi_t)$.*

(2) *If $p \in \partial\mathbb{D}$ is the Denjoy – Wolff point of (ϕ_t) then the non-tangential limit $f(p) = \angle \lim_{z \to p} f(z)$ exists and $f(p) \in \partial\mathbb{D}$ is the Denjoy – Wolff point of (φ_t).*

Proof. Let $(\Omega_1 = I_1 \times \mathbb{R}, h_1, z \mapsto z + it)$ be the holomorphic model of (ϕ_t) and let $(\Omega_2 = I_2 \times \mathbb{R}, h_2, z \mapsto z + it)$ be the holomorphic model of (φ_t), where I_1, I_2 are (possibly unbounded) open intervals in $\mathbb{R}$.

Assume M is the exceptional maximal contact arc for (ϕ_t) such that p is its initial point. By Proposition 4.8, $I \neq \mathbb{R}$ and we can assume without loss of generality that $I_1 = (0, \rho)$, with $-\infty < 0 < \rho \leq +\infty$ and that $h(M) = \{it, t \in \mathbb{R}\}$.

Let C be the connected component of $A(\phi_t)$ which corresponds to p (see Proposition 4.9). Thus there exists $0 < b \leq \rho$ such that $\overline{C} = \{\zeta \in \mathbb{C} : 0 \leq \mathsf{Re}\,\zeta \leq b\}$. Since we are assuming that p is not the Denjoy – Wolff point of (ϕ_t), from Proposition 4.9, it follows that $b < +\infty$. Note that $b = \rho$ if and only if $h(\mathbb{D}) = \Omega_1$, that is, (ϕ_t) is a (hyperbolic) group of automorphisms.

By Lemma 5.1, $f = h_2^{-1} \circ \tau \circ h_1$, with $\tau(z) = u(\mathsf{Re}\,z) + i(\mathsf{Im}\,z + v(\mathsf{Re}\,z))$, where $u : I_1 \to I_2$ is a homeomorphism. With no loss of generality we can assume that u is orientation preserving, and we let $\tilde{a} = \lim_{x \to 0^+} u(x)$ and $\tilde{b} = \lim_{x \to b^-} u(x)$.

As remarked in the proof of Proposition 6.2, $\tilde{C} := \tau(C)$ is a connected component in $A(\varphi_t)$. Clearly, $\overline{\tilde{C}} = \{\zeta \in \mathbb{C} : \tilde{a} \leq \mathsf{Re}\,\zeta \leq \tilde{b}\}$. Let $p(\tilde{C})$ denote the associated boundary regular fixed point for (φ_t).

Let $\gamma : [0, 1) \to \mathbb{D}$ be a continuous curve which converges non-tangentially to p. By [8, Lemma 4.4], there exist $a < a' < b' < b$ and $t_0 < 1$ such that $a' \leq \mathsf{Re}\,h_1(\gamma(t)) \leq b'$ for all $t \in (t_0, 1)$. Therefore, there exist $\tilde{a} < \tilde{a}' < \tilde{b}' < \tilde{b}$ such that $\tilde{a}' \leq \mathsf{Re}\,\tau(\mathsf{Re}\,(h_1(\gamma(t)))) \leq \tilde{b}'$ for all $t \in (t_0, 1)$. Therefore, the continuous curve $\tau \circ h_1 \circ \gamma$ is all contained in the strip $\{\zeta \in \mathbb{C} : \tilde{a}' \leq \mathsf{Re}\,\zeta \leq \tilde{b}'\}$ and its imaginary part converges to $-\infty$. It follows then from [8, Prop. 4.5] that

$$\lim_{t \to 1} f(\gamma(t)) = \lim_{t \to 1} h_2^{-1}(\tau(h_1(\gamma(t))) = p(\tilde{C}).$$

So, there exists $\angle\lim_{z\to p} f(z)$ and its value is $p(\tilde{C})$, which is a boundary regular fixed point for (ϕ_t).

(2) In case p is the Denjoy–Wolff point of (ϕ_t), the argument is similar and we omit the proof. $\qquad\square$

Remark 7.2. From the previous proof it follows that if I_2 is relatively compact in $\mathbb{R}$, then $\overline{\tilde{C}} = \{\zeta \in \mathbb{C} : \tilde{a} \le \mathsf{Re}\,\zeta \le \tilde{b}\}$, with $|\tilde{a}|, |\tilde{b}| < +\infty$. Therefore, by Proposition 4.9, in this case $f(p)$ is not the Denjoy–Wolff point of (φ_t). That is, $f(p)$ can be the Denjoy–Wolff point of (φ_t) only if the semigroup is parabolic.

In the following examples we show that the unrestricted limit of the homeomorphism f might no exist at the Denjoy–Wolff point or at boundary regular fixed points which start an exceptional maximal contact arc.

Example 7.3 (*Non existence of unrestricted limit at Denjoy–Wolff point*)**.** Let $Q = \{z \in \mathbb{S} : \mathsf{Im}\,z\,\mathsf{Re}\,z > 1\}$ and let $h : \mathbb{D} \to Q$ be a Riemann map. Consider the semigroup (ϕ_t) defined by $\phi_t(z) := h^{-1}(h(z)+it)$, $t \ge 0$. The point $p = \lim_{t\to+\infty} h^{-1}(\frac{1}{2}+it)$ is the Denjoy–Wolff point of (ϕ_t). Let $v : (0,1) \to \mathbb{R}$ be defined by $v(x) = -1/x$ for all $x \in (0,1)$, and define $\tau(z) := \mathsf{Re}\,z + i(\mathsf{Im}\,z + v(\mathsf{Re}\,z))$. Write $\tilde{Q} = \tau(Q) = \{z \in \mathbb{S} : \mathsf{Im}\,z > 0\}$ and let $\tilde{h} : \mathbb{D} \to \tilde{Q}$ be a Riemann map. The semigroup (φ_t) defined by $\varphi_t(z) := \tilde{h}^{-1}(\tilde{h}(z) + it)$, $t \ge 0$ is topologically conjugated to (ϕ_t) via $f := \tilde{h}^{-1}\circ\tau\circ h$. We claim that the unrestricted limit of f at p does not exist. The point $q := \angle\lim_{z\to p} f(z)$ is the Denjoy–Wolff point of (φ_t). Notice that h and $\tilde{h}$ can be extended to homeomorphisms of $\overline{\mathbb{D}}$ onto $\overline{Q}$ and $\overline{\tilde{Q}}$, respectively. We also denote by h and $\tilde{h}$ those extensions. Take $z_n = h^{-1}\left(\frac{1}{n} + (1+n)i\right) \in \mathbb{D}$. By the properties of h, we deduce that the sequence (z_n) converges to p. Moreover,

$$f(z_n) = \tilde{h}^{-1}(\tau(h(z_n))) = \tilde{h}^{-1}\left(\frac{1}{n} + i\right) \to \tilde{h}^{-1}(i) \ne q.$$

Example 7.4 (*Non existence of unrestricted limit at boundary regular fixed points starting an exceptional maximal contact arc*)**.** Let $h : \mathbb{D} \to \mathbb{S}$ be the Riemann map given by $h(z) = \frac{1}{2} + \frac{i}{\pi}\log\frac{1+z}{1-z}$. Let (ϕ_t) be the (semi)group of hyperbolic automorphisms of $\mathbb{D}$ defined by $\phi_t(z) := h^{-1}(h(z)+it)$, $t \ge 0$. Then $1 = \lim_{\mathsf{Im}\,z\to+\infty} h^{-1}(z)$ is the Denjoy–Wolff point of (ϕ_t), while $-1 = \lim_{\mathsf{Im}\,z\to-\infty} h^{-1}(z)$ is a boundary regular fixed point of (ϕ_t). Let $\tau : \mathbb{S} \to \mathbb{S}$ the homeomorphism defined by $\tau(z) = \mathsf{Re}\,z + i(\mathsf{Im}\,z + \frac{1}{\mathsf{Re}\,z})$. Let $f := h^{-1}\circ\tau\circ h$ be a homeomorphism of $\mathbb{D}$. Then $f\circ\phi_t = \phi_t\circ f$. By Proposition 7.1, f has non-tangential limit at -1 and it is easy to see that $\angle\lim_{z\to-1} f(z) = -1$, for instance by taking the limit of f along the (non-tangential) curve $(0,+\infty) \ni s \mapsto h^{-1}(1/2-s)$ converging to -1.

Consider now the curve $\gamma : (0,1) \to \mathbb{S}$ defined by $\gamma(s) := (1-s) + i(s-1)^{-1}$. Then $h^{-1}(\gamma(s))$ is a curve in $\mathbb{D}$ which converges (tangentially) to -1 as $s \to 1$. However,

$$\lim_{s\to 1} f(h^{-1}(\gamma(s))) = \lim_{s\to 1} h^{-1}(\tau(\gamma(s))) = \lim_{s\to 1} h^{-1}(1-s) = h^{-1}(0) \ne -1.$$

Therefore, f is not continuous at -1.

As we already proved in Proposition 3.5, every non-elliptic semigroup of holomorphic self-maps can be topologically conjugated to a hyperbolic one (with $\mathbb{S}$ as model domain). Reversing this implication, it follows that every non-elliptic semigroup can be topologically conjugated to one whose model domain is $\mathbb{R} \times \mathbb{R}$ (that is, a parabolic semigroup of zero hyperbolic step). In this case, there exist no exceptional maximal contact arcs, which show that exceptional maximal contact arcs are not topological invariants.

However, if one stays in the class of hyperbolic semigroups, the question whether an exceptional maximal contact arc is a topological invariant is less trivial, as the following example shows:

Example 7.5. Let $\Omega = \{z \in \mathbb{S} : \mathsf{Im}\, z\, \mathsf{Re}\, z > -1\}$ and let $h : \mathbb{D} \to \Omega$ be a Riemann map. For $t \geq 0$, let $\varphi_t(z) := h^{-1}(h(z) + it)$. It is easy to see that (φ_t) is a semigroup of holomorphic self-maps of $\mathbb{D}$. The point $p = \lim_{t \to -\infty} h^{-1}(t - i/2t)$ is a boundary super repelling fixed point of third type for this semigroup. Let $v : (0,1) \to \mathbb{R}$ be a continuous function and define $\tau(z) := \mathsf{Re}\, z + i(\mathsf{Im}\, z + v(\mathsf{Re}\, z))$. Write $\Omega_v := \tau(\Omega) = \{z \in \mathbb{S} : \mathsf{Im}\, z > -\frac{1}{\mathsf{Re}\, z} + v(\mathsf{Re}\, z)\}$. Let $h_v : \mathbb{D} \to \Omega_v$ be a Riemann map. Let $\phi_t(z) := h_v^{-1}(h_v(z) + it)$, $t \geq 0$. The semigroup (ϕ_t) is clearly topologically conjugated to (φ_t). On the one hand, taking $v(x) := 2/x$, we obtain a semigroup whose unique fixed point is its Denjoy–Wolff point and the map $f = h_v^{-1} \circ \tau \circ h$ sends the exceptional maximal contact arc starting at p to the Denjoy–Wolff point of (ϕ_t). In particular, the boundary super repelling fixed point of third type p is sent to the Denjoy–Wolff point. On the other hand, taking $v(x) = 1/x$ we obtain a semigroup whose unique fixed point is its Denjoy–Wolff point and the map $f = h_v^{-1} \circ \tau \circ h$ sends the exceptional maximal contact arc starting at p onto an exceptional maximal contact arc for (ϕ_t) having a non-fixed point as initial point. In particular, the boundary super repelling fixed point of third type p is sent to a contact, not fixed, point.

Proposition 7.6. *Let (ϕ_t) be a hyperbolic semigroup of holomorphic self-maps of $\mathbb{D}$ with Denjoy–Wolff point $\sigma \in \partial\mathbb{D}$. Let M be an exceptional maximal contact arc for (ϕ_t) whose initial point is $x_0 \in \partial\mathbb{D}$.*

  (1) *if x_0 is a regular boundary fixed point for (ϕ_t) and (φ_t) is a hyperbolic semigroup of holomorphic self-maps of $\mathbb{D}$, topologically conjugated to (ϕ_t) via the homeomorphism $f : \mathbb{D} \to \mathbb{D}$ then (φ_t) has an exceptional maximal contact arc whose initial point is $\angle \lim_{z \to x_0} f(z)$.*

  (2) *If x_0 is not a boundary regular fixed point for (ϕ_t) then there exists a homeomorphism $f : \mathbb{D} \to \mathbb{D}$ such that $(\varphi_t := f^{-1} \circ \phi_t \circ f)$ is a hyperbolic semigroup of holomorphic self-maps of $\mathbb{D}$ and $\lim_{z \to p} f(z) = \sigma$ for all $p \in M$.*

Proof. (1) By Remark 7.2, $f(x_0) := \angle \lim_{z \to x_0} f(z)$ is a boundary regular fixed point of (φ_t) different from the Denjoy–Wolff point. If $f(x_0)$ does not belong to the closure of an exceptional maximal contact arc for (φ_t), by Proposition 5.2, f^{-1} has unrestricted limit at $f(x_0)$ and

$$x_0 = \lim_{(0,1) \ni r \to 1} f^{-1}(f(rx_0)) = f^{-1}(f(x_0)) \notin E(\phi_t),$$

a contradiction. Therefore $f(x_0)$ belongs to the closure of an exceptional maximal contact arc for (φ_t) and it is in fact its initial point, being a fixed point.

(2) Let $(\Omega = (0, b) \times \mathbb{R}, h, z \mapsto z + ti)$, $0 < b < +\infty$ be the holomorphic model of (ϕ_t) and let $Q := h(\mathbb{D})$. Let M be an exceptional maximal contact arc with initial point x_0. By Theorem 4.4, we can assume without loss of generality that $\operatorname{Re} h(z) = 0$ for all $z \in M$.

Assume x_0 is not a boundary *regular* fixed point for (ϕ_t).

If x_0 is a super-repelling boundary fixed point for (ϕ_t), let

$$\lambda_n = \sup\{y : s + iy \notin Q, s \in (0, 1/n]\}.$$

Since x_0 is super-repelling, by Theorem 4.13, $\lim_{r \to 1} \operatorname{Re} h(rx_0) = 0$ and $\lim_{r \to 1} \operatorname{Im} h(rx_0) = -\infty$. Thus λ_n tends to $-\infty$. For each $n \in \mathbb{N}$, take $0 < s_n \leq 1/n$ and y_n such that $s_n + iy_n \notin Q$ and $y_n \geq \lambda_n - 1/n$. Take a monotone subsequence (s_{n_k}) of (s_n) and a continuous function $v : I \to I$ such that $v(s_{n_k}) = -2\lambda_{n_k}$. Define $\tau : \Omega \to \Omega$ by $\tau(x + iy) := x + i(y + v(x))$. Note that τ is a homeomorphism. Let $h_2 : \mathbb{D} \to \tau(Q)$ be a Riemann map. Let (φ_t) be the semigroup in $\mathbb{D}$ defined by $\varphi_t(z) := h_2^{-1}(h_2(z) + it)$. Clearly, (φ_t) is topologically conjugated to (ϕ_t). Since $w_n = \tau(s_n + iy_n) \notin \tau(Q)$ and

$$\lim_k \operatorname{Im}(w_{n_k}) = \lim_k (y_{n_k} + v(s_{n_k})) \geq \lim_k (-\lambda_{n_k} + 2/n_k) = +\infty,$$

it follows that f maps the exceptional maximal contact arc M to the Denjoy–Wolff point of (φ_t).

Next, suppose that x_0 is not a boundary fixed point. Then, by Theorem 4.13, there exists $\lim_{r \to 1} h(rx_0) = iy_0$ with $y_0 \in \mathbb{R}$. Therefore there exist points $x_n + iy_n \in \Omega \setminus Q$ such x_n goes to 0 and y_n goes to y_0. Take a continuous function $v : I \to I$ such that $\lim_{x \to 0} v(x) = +\infty$. As before, define $\tau : \Omega \to \Omega$ by $\tau(x + iy) := x + i(y + v(x))$ and let $h_2 : \mathbb{D} \to \tau(Q)$ be any Riemann map. By construction, $\tau(x_n + iy_n) \in \Omega \setminus h_2(\mathbb{D})$ and $\operatorname{Im} \tau(x_n + iy_n)$ goes to $+\infty$. Therefore the semigroup (φ_t) defined by $\varphi_t(z) := h_2^{-1}(h_2(z) + it)$ for $t \geq 0$ is topologically conjugated to (ϕ_t) and f maps the exceptional maximal contact arc W to the Denjoy–Wolff point of (φ_t). $\qquad\square$

The ω-limit of f at an exceptional maximal contact arc is described by the following proposition:

Proposition 7.7. *Let (ϕ_t) and (φ_t) be two non-elliptic semigroups in $\mathbb{D}$ with holomorphic models $(\Omega_1 = I_1 + i\mathbb{R}, h_1, z \mapsto z + ti)$ and $(\Omega_2 = I_2 + i\mathbb{R}, h_2, z \mapsto z + ti)$, respectively. Suppose (ϕ_t) and (φ_t) are topologically conjugated via the homeomorphism $f : \mathbb{D} \to \mathbb{D}$. Let p and q be the Denjoy–Wolff points of (ϕ_t) and (φ_t), respectively. Assume that W is an exceptional maximal contact arc for (ϕ_t). Denote by $S = \{z \in \mathbb{D} \setminus \{0\} : z/|z| \in W\}$ and let x_0 be the initial point of W. Then the set*

$$E(W) = \left\{ w \in \overline{\mathbb{D}} : \exists \{z_n\} \subset S, z_n \to z \in \overline{W}, f(z_n) \to w \right\}$$

is a compact connected arc in $\partial\mathbb{D}$ containing q which is contained in an exceptional maximal contact arc for (φ_t).

Proof. Let $Q_1 := h_1(\mathbb{D})$ and let $Q_2 := h_2(\mathbb{D})$. By Lemma 5.1, (ϕ_t) and (φ_t) are topologically conjugated if and only if there exists a homeomorphism $\tau : \Omega_1 \to \Omega_2$ given by (5.1) and $\tau(Q_1) = Q_2$. By Proposition 3.3, $f = h_2^{-1} \circ \tau \circ h_1$.

Since $\angle \lim_{z \to p} f(z) = q$ by Proposition 7.1, it follows that $q \in E(W)$. Moreover, it is easy to see that $E(W)$ is a compact subset of $\partial \mathbb{D}$. Therefore we are left to check that $E(W)$ is connected. Indeed, assume this is not the case. Then there exist two compact sets A and B such that $E(W) = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. Write $k = d(A, B) > 0$. For $j = 1, 2$, take $w_j \in A_j$, $z_{n,j} \in \mathbb{D}$, for all n, with $\frac{z_{n,j}}{|z_{n,j}|} \in \overline{W}$, $z_{n,j} \to z_j \in \overline{W}$ and $f(z_{n,j}) \to w_j$. We may assume that $d(f(z_{n,j}), w_j) < k/4$. Let C_n be the arc in $\overline{W}$ that joins $\frac{z_{n,1}}{|z_{n,1}|}$ with $\frac{z_{n,2}}{|z_{n,2}|}$ and

$$\Gamma_n = [z_{n,1}, r_n z_{n,1}] \cup r_n C_n \cup [z_{n,2}, r_n z_{n,2}]$$

where $r_n = \max\{|z_{n,1}|, |z_{n,2}|\}$. Notice that Γ_n is connected. Consider the continuous function $l : \Gamma_n \to \mathbb{R}$ given by $l(z) = d(f(z), A_1)$. Since $l(z_{n,1}) < k/4$ and $l(z_{n,2}) \geq d(w_2, A_1) - d(z_{n,2}, w_2) \geq k - k/4 = 3k/4$. So there is $\alpha_n \in \Gamma_n$ such that $d(f(\alpha_n), A_1) = k/2$. Since $|\alpha_n| \geq \min\{|z_{n,1}|, |z_{n,2}|\}$, we can take a subsequence such that $\alpha_{n_k} \to z \in W$ and $f(\alpha_{n_k}) \to w$. Clearly, $w \in E(W)$ and $d(f(w), A_1) = k/2$. A contradiction. Hence $E(W)$ is connected. Clearly, $E(W)$ is contained in an exceptional maximal contact arc for (φ_t), for otherwise $f^{?1}$ would map points of $\partial \mathbb{D} \setminus E(\varphi_t)$ into $E(\phi_t)$. $\square$

Example 7.8. Let $\Omega_1 = \{z \in \mathbb{S} : \text{Im } z > 0\}$ and $h_1 : \mathbb{D} \to \Omega_1$ a Riemann map. Consider the semigroup (ϕ_t) defined by $\phi_t(z) := h_1^{-1}(h_1(z) + it)$, $t \geq 0$. The arc $W = h_1^{-1}([0, \infty)i)$ is an exceptional maximal contact arc for (ϕ_t). Define $\tau : \mathbb{S} \to \mathbb{S}$ as $\tau(x + iy) := x + i(y - 1/x)$, $\Omega_2 := \tau(\Omega_1) = \{z \in \mathbb{S} : \text{Im } z \, \text{Re } z > -1\}$ and let $h_2 : \mathbb{D} \to \Omega_2$ be a Riemann map. The semigroup (φ_t) defined by $\varphi_t(z) := h_2^{-1}(h_2(z) + it)$, $t \geq 0$, is topological conjugated to (ϕ_t) via the homeomorphism $f = h_2^{-1} \circ \tau \circ h_1$. The semigroup (φ_t) has an exceptional maximal contact arc $\tilde{W} = h_2^{-1}(\mathbb{R}i)$ with initial point a fixed point x_0. Notice that for all $w \in W$ it holds $x_0 = \lim_{z \to w} f(z)$, namely, the map f sends the arc W to the point x_0. This does not contradict the previous proposition, since $\tilde{W}$ is the ω-limit of f at the Denjoy–Wolff point of (ϕ_t) .

8. Elliptic case

In this final section we show how to recover the results of Sections 5 and 6 in case of elliptic semigroups which are not groups. The key is to replace Lemma 5.1 by the following:

Lemma 8.1. *Let (ϕ_t) and (φ_t) be two elliptic semigroups in $\mathbb{D}$, which are not groups. Let $(\mathbb{C}, h_1, z \mapsto e^{\lambda_1 t} z)$ be a holomorphic model of (ϕ_t) and let $(\mathbb{C}, h_2, z \mapsto e^{\lambda_2 t} z)$ be a holomorphic model of (φ_t). Then, (ϕ_t) and (φ_t) are topologically conjugated if and only if there exist a homeomorphism u of the unit circle $\partial \mathbb{D}$ and a continuous map $v : \partial \mathbb{D} \to (0, +\infty)$*

such that $\tau := \theta_{\lambda_2}^{-1} \circ \tau_0 \circ \theta_{\lambda_1}$ satisfies $\tau(h_1(\mathbb{D})) = h_2(\mathbb{D})$, where

$$\tau_0(z) := \begin{cases} 0, & z = 0 \\ |z| u(z/|z|) v(z/|z|), & z \neq 0 \end{cases}, \ z \in \mathbb{C},$$

and, given $\lambda = a + ib$ with $a < 0$,

$$\theta_\lambda(z) := z|z|^{-(1+1/a)} \exp\left(-i\frac{b}{a}\mathrm{Log}|z|\right), z \in \mathbb{C}^* \ and \ \theta_\lambda(0) := 0.$$

As already remarked, if the semigroup (ϕ_t) is elliptic, the set $E(\phi_t) = \emptyset$. Using the previous lemma and mimicking the proof of Proposition 5.2, one can prove the following extension result:

Proposition 8.2. *Let (ϕ_t) and (φ_t) be two elliptic semigroups of holomorphic self-maps of $\mathbb{D}$, which are not groups. Suppose (ϕ_t) and (φ_t) are topologically conjugated via the homeomorphism $f : \mathbb{D} \to \mathbb{D}$. Then f extends to a homeomorphism*

$$f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}.$$

Moreover, for all $p \in \partial\mathbb{D}$ it holds $T(p) = T(f(p))$ and $f(\phi_t(p)) = \varphi_t(f(p))$ for all $t \geq 0$.

And also,

Proposition 8.3. *Let (ϕ_t) and (φ_t) be two elliptic semigroups in $\mathbb{D}$, which are not groups. Suppose (ϕ_t) and (φ_t) are topologically conjugated via the homeomorphism $f : \mathbb{D} \to \mathbb{D}$.*

(1) *If W is a maximal contact arc for (ϕ_t), then f extends to a homeomorphism from W onto $f(W)$ and $f(W)$ is a maximal contact arc for (φ_t).*

(2) *Let $p \in \partial\mathbb{D}$ be a boundary fixed point for (ϕ_t). Then the unrestricted limit*

$$f(p) := \lim_{z \to p} f(z) \in \partial\mathbb{D}$$

exists and $f(p) \in \partial\mathbb{D}$ is a boundary fixed point for (φ_t). Moreover,

(a) *if p is a boundary regular fixed point for (ϕ_t) then $f(p)$ is a boundary regular fixed point for (φ_t),*

(b) *if p is a boundary super-repelling fixed point of first type (respectively of second type) for (ϕ_t) then $f(p)$ is a a boundary super-repelling fixed point of first type (respectively of second type) for (φ_t).*

We end this section showing that Proposition 8.2 is no longer true for groups of elliptic automorphisms.

Example 8.4. Consider the group of automorphisms (ϕ_t) where $\phi_t(z) = e^{it}z$ for all $t \in \mathbb{R}$ and $z \in \mathbb{D}$ and the continuous function $f : \mathbb{D} \to \mathbb{D}$ given by $f(z) = z\exp(i\ln(1 - |z|))$ for all $z \in \mathbb{D}$. It is clear that f is an homeomorphism of the unit disc with inverse function $f^{-1}(z) = z\exp(-i\ln(1 - |z|))$ and

$$f(\phi_t(z)) = \phi_t(f(z)) \quad t \geq 0, \ z \in \mathbb{D}.$$

But f has no continuous extension at any point of $\partial\mathbb{D}$.

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